

# Directed Width Parameters and Circumference of Digraphs

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January 14, 2014

## Abstract

We prove that the directed treewidth, DAG-width and Kelly-width of a digraph are bounded above by its circumference plus one.

**Keywords:** arboreal decomposition, directed treewidth, DAG-decomposition, DAG-width, Kelly decomposition, Kelly-width.

## 1 Introduction

The *circumference* of an undirected graph (resp. digraph)  $G$ , denoted by  $\text{circ}(G)$ , is the length of a longest simple undirected (resp. directed) cycle in  $G$ . The circumference of a DAG is defined to be one. The circumference of an undirected tree is defined to be two. Birmele [Bir03] proved that the treewidth of an undirected graph  $G$ , denoted by  $\text{tw}(G)$ , is at most its circumference minus one.

**Theorem 1.** (Birmele [Bir03]) For an undirected graph  $G$ ,  $\text{tw}(G) \leq \text{circ}(G) - 1$ .

Motivated by the success of treewidth in algorithmic and structural graph theory, efforts have been made to generalize treewidth to digraphs. Johnson et al. [JRST01] introduced the first directed analogue of treewidth called directed treewidth. Berwanger et al. [BDHK06] and independently Obdrzalek [Obd06] introduced DAG-width. Hunter and Kreutzer [HK08] introduced Kelly-width. For a digraph  $G$ , let  $\text{dtw}(G)$ ,  $\text{dgw}(G)$  and  $\text{kw}(G)$  denote its directed treewidth, DAG-width and Kelly-width respectively. All these *directed* width measures are generalizations of undirected treewidth i.e., for an *undirected* graph  $G$ , let  $\overleftrightarrow{G}$  be the *digraph* obtained by replacing each edge  $\{u, v\}$  of  $G$  by two directed edges  $(u, v)$  and  $(v, u)$ , then:

- $\text{dtw}(\overleftrightarrow{G}) = \text{tw}(G)$  [JRST01, Theorem 2.1]
- $\text{dgw}(\overleftrightarrow{G}) = \text{tw}(G) + 1$  [BDHK06, Proposition 5.2]
- $\text{kw}(\overleftrightarrow{G}) = \text{tw}(G) + 1$  [HK08]

We prove that the directed treewidth, DAG-width and Kelly-width of a digraph are bounded above by its circumference plus one. Our proofs generalize Birmele’s idea of constructing a tree decomposition using a depth-first search tree. For the directed treewidth we construct an arboreal decomposition from the depth-first search tree very naturally. The underlying arborescence is the depth-first search tree itself. For the DAG-width and Kelly-width we construct the underlying DAG using the depth-first search tree and some carefully chosen additional edges. Constructing the corresponding “bags” requires some additional work to satisfy the strict guarding conditions of DAG-decompositions and Kelly-decompositions. Our main theorem is as follows:

**Theorem 2.** For a digraph  $G$ ,

- $\text{dtw}(G) \leq \text{circ}(G) + 1$
- $\text{dgw}(G) \leq \text{circ}(G) + 1$
- $\text{kw}(G) \leq \text{circ}(G) + 1$

Birmele’s theorem is tight as  $\text{tw}(K_n) = n - 1$  and  $\text{circ}(K_n) = n$ . Since  $\text{dtw}(\overleftrightarrow{K_n}) = n - 1$ ,  $\text{dgw}(\overleftrightarrow{K_n}) = n$  and  $\text{kw}(\overleftrightarrow{K_n}) = n$ , we conjecture that [Theorem 2](#) can be improved with the following tight bounds:

**Conjecture 3.** For a digraph  $G$ ,

- $\text{dtw}(G) \leq \text{circ}(G) - 1$
- $\text{dgw}(G) \leq \text{circ}(G)$
- $\text{kw}(G) \leq \text{circ}(G)$

Birmele’s theorem does not hold for pathwidth since complete binary trees have unbounded pathwidth. Nesetril and Ossona de Mendez [\[NdM12\]](#) showed that the pathwidth of a 2-connected graph  $G$  is at most  $(\text{circ}(G) - 2)^2$ . Marshall and Wood [\[MW13\]](#) improved this bound to  $\lfloor \text{circ}(G)/2 \rfloor (\text{circ}(G) - 1)$ . Generalizing these results to directed pathwidth, under a suitable *directed connectivity* assumption is an interesting open problem.

## 1.1 Notation

We use standard graph theory notation and terminology (see [\[Die05\]](#)). All digraphs are finite and simple (i.e. no self loops and no multiple arcs). For a digraph  $G$ , we write  $V(G)$  for its vertex set and  $E(G)$  for its arc set. For  $S \subseteq V(G)$  we write  $G[S]$  for the subdigraph induced by  $S$ , and  $G \setminus S$  for the subdigraph induced by  $V(G) - S$ .

We use the term DAG when referring to directed acyclic graphs. A node is a *root* if it has no incoming arcs. The DAG  $T$  is an *arborescence* if it has a unique root  $r$  such that for every node  $i \in V(T)$  there is a unique directed walk from  $r$  to  $i$ . Note that every arborescence arises from an undirected tree by selecting a root and directing all edges away from the root.

Let  $T$  be a DAG. For two *distinct* nodes  $i$  and  $j$  of  $T$ , we write  $i \prec_T j$  if there is a directed walk in  $T$  with first node  $i$  and last node  $j$ . For convenience, we write  $i \prec j$  whenever  $T$  is clear from the context. For nodes  $i$  and  $j$  of  $T$ , we write  $i \preceq j$  if either  $i = j$  or  $i \prec j$ . For an arc  $e = (i, j)$

and a node  $k$  of  $T$ , we write  $e \prec k$  if either  $j = k$  or  $j \prec k$ . We write  $e \sim i$  (and  $e \sim j$ ) to mean that  $e$  is incident with  $i$  (and  $j$  respectively). We define  $T_{\succeq v} = T[\{x \mid x \succeq v\}]$ .

Let  $\mathcal{W} = (W_i)_{i \in V(T)}$  be a family of finite sets called *node bags*, which associates each node  $i$  of  $T$  to a node bag  $W_i$ . We write  $W_{\succeq i}$  to denote  $\bigcup_{j \succeq i} W_j$ . For an arc  $e$  of  $T$ , we write  $W_{\succ e}$  to denote  $\bigcup_{j \succ e} W_j$ . Let  $\mathcal{A} = (A_e)_{e \in E(T)}$  be a family of finite sets called *arc bags*, which associates each arc  $e$  of  $T$  to an arc bag  $A_e$ . We write  $A_{\sim i}$  to denote  $\bigcup_{e \sim i} A_e$ .

## 1.2 Guarding, $X$ -normal and Directed unions

Width measures like DAG-width and Kelly-width are based on the following notion of *guarding*:

**Definition 4.** [Guarding] Let  $G$  be a digraph and  $W, X \subseteq V(G)$ . We say  $X$  *guards*  $W$  if  $W \cap X = \emptyset$ , and for all  $(u, v) \in E(G)$ , if  $u \in W$  then  $v \in W \cup X$ .

In other words,  $X$  guards  $W$  means that there is no directed path in  $G \setminus X$  that starts from  $W$  and leaves  $W$ . The notion of directed treewidth is based on a weaker condition:

**Definition 5.** [ $X$ -normal] Let  $G$  be a digraph and  $W, X \subseteq V(G)$ . We say  $W$  is  *$X$ -normal* if  $W \cap X = \emptyset$ , and there is no directed path in  $G \setminus X$  with first and last vertices in  $W$  that uses a vertex of  $G \setminus (W \cup X)$ .

In other words,  $W$  is  $X$ -normal means that there is no directed path in  $G \setminus X$  that starts from  $W$ , leaves  $W$  and then returns to  $W$ . A digraph  $D$  is a *directed union* of digraphs  $D_1$  and  $D_2$  if  $D_1$  and  $D_2$  are induced subgraphs of  $D$ ,  $V(D_1) \cup V(D_2) = V(D)$ , and no edge of  $D$  has head in  $V(D_1)$  and tail in  $V(D_2)$ . The directed treewidth, DAG-width and Kelly-width are closed under directed unions (see [JRST01, BDH<sup>+</sup>12, MTV10]). The following theorem is immediate.

**Theorem 6.** ([JRST01, BDH<sup>+</sup>12, MTV10]) The directed treewidth (resp. DAG-width, Kelly-width) of a digraph  $G$  is equal to the maximum directed treewidth (resp. DAG-width, Kelly-width) taken over the strongly-connected components of  $G$ .

Also, the circumference of a digraph  $G$  is equal to the maximum circumference taken over the strongly-connected components of  $G$ . Hence, we may assume that all digraphs are strongly-connected in the rest of this paper.

## 1.3 Depth-first search tree

Let  $G$  be a strongly-connected digraph. Let  $T$  be a depth-first search tree of  $G$  starting at an arbitrary root  $r \in V(G)$ . The tree  $T$  is an arborescence rooted at  $r$ . The edges of  $G$  are classified into one of the four types : *tree edges*, *forward edges*, *back edges* and *cross edges* (see [CLRS01]). For a vertex  $v \in V(G)$ , let  $dfs(v)$  be the “dfs number” of  $v$  i.e., the time-stamp assigned to  $v$  when  $v$  is visited for the first time during the construction of  $T$ .

## 2 Directed treewidth and Circumference

**Definition 7.** [Arboreal decomposition and directed treewidth[JIRST01]] An *arboreal decomposition* of a digraph  $G$  is a triple  $\mathcal{D} = (T, \mathcal{W}, \mathcal{A})$ , where  $T$  is an arborescence, and  $\mathcal{W} = (W_i)_{i \in V(T)}$  is a family of subsets (node bags) of  $V(G)$ , and  $\mathcal{A} = (A_e)_{e \in E(T)}$  is a family of subsets (arc bags) of  $V(G)$ , such that:

- $\mathcal{W}$  is a partition of  $V(G)$ . (DTW-1)

- For each arc  $e \in E(T)$ ,  $W_{\succ_e}$  is  $A_e$ -normal. (DTW-2)

The width of an arboreal decomposition  $\mathcal{D} = (T, \mathcal{W}, \mathcal{A})$  is defined as  $\max\{|W_i \cup A_{\sim i}| : i \in V(T)\} - 1$ . The *directed treewidth* of  $G$ , denoted by  $\text{dtw}(G)$ , is the minimum width over all possible arboreal decompositions of  $G$ .

**Theorem 8.** For a digraph  $G$ ,  $\text{dtw}(G) \leq \text{circ}(G) + 1$ .

*Proof.* Let  $T$  be the depth-first search tree constructed in Section 1.3. Let  $\mathcal{W} = (W_i)_{i \in V(T)}$  be a partition of  $V(G)$  defined as  $W_i = \{i\}$  for each  $i \in V(T)$ . For every edge  $e = (r, v) \in E(T)$ , we define  $A_e = \{r\}$ . For every edge  $e = (u, v) \in E(T)$  such that  $u \neq r$  we define  $A_e$  as follows:

- if there are no back edges from  $W_{\succ_e}$ , we define  $A_e = \{r\}$ .
- if there are back edges from  $W_{\succ_e}$ , let  $B$  be the set of all vertices  $b \preceq u$  such that there is a back edge from some vertex in  $W_{\succ_e}$  to  $b$ . Let  $b_0$  be the minimal element in  $B$  with respect to  $\preceq$ . Let  $A_e = \{r\} \cup \{x \mid b_0 \preceq x \preceq u\}$ . Note that  $|\{x \mid b_0 \preceq x \preceq u\}| \leq l - 1$  and hence  $|A_e| \leq l$ .

Let  $\mathcal{A} = (A_e)_{e \in E(T)}$ . We claim that  $\mathcal{D} = (T, \mathcal{W}, \mathcal{A})$  is an arboreal decomposition of  $G$  of width at most  $l + 1$ . By construction,  $\mathcal{W} = (W_i)_{i \in V(T)}$  is a partition of  $V(G)$  so  $\mathcal{D}$  satisfies (DTW-1). To show that  $\mathcal{D}$  satisfies (DTW-2) we must show that for each arc  $e \in E(T)$ ,  $W_{\succ_e}$  is  $A_e$ -normal. For every edge  $e = (r, v) \in E(T)$ , every directed path that leaves  $W_{\succ_e}$  and returns to  $W_{\succ_e}$  must go through the root  $r$ . Hence,  $W_{\succ_e}$  is  $A_e$ -normal. For every edge  $e = (u, v) \in E(T)$  such that  $u \neq r$  we consider the following cases:

- if there are no back edges from  $W_{\succ_e}$ , every directed path that leaves  $W_{\succ_e}$  and returns to  $W_{\succ_e}$  must go through the root  $r$ . Hence,  $W_{\succ_e}$  is  $A_e$ -normal.
- if there are back edges from  $W_{\succ_e}$ , every directed path that leaves  $W_{\succ_e}$  and returns to  $W_{\succ_e}$  must go through the root  $r$  (or) go through a vertex in  $\{x \mid b_0 \preceq x \preceq u\}$ . Hence,  $W_{\succ_e}$  is  $A_e$ -normal.

The size of each arc bag is at most  $l$ . Let  $e_1 = (u, v), e_2 = (v, w) \in E(T)$ . Let  $B_1$  be the set of all vertices  $b \preceq u$  such that there is a back edge from some vertex in  $W_{\succ_{e_1}}$  to  $b$ . Let  $B_2$  be the set of all vertices  $b' \preceq v$  such that there is a back edge from some vertex in  $W_{\succ_{e_2}}$  to  $b'$ . Note that  $B_2 \subseteq B_1 \cup \{v\}$ . Hence, for every  $i \in V(T)$  the number of vertices in  $A_{\sim i}$  is at most  $l + 1$  and the number of vertices in  $W_i \cup A_{\sim i}$  is at most  $l + 2$ . Hence, the width of  $\mathcal{D} = (T, \mathcal{W}, \mathcal{A})$  is at most  $l + 1$ . □

### 3 DAG-width and Circumference

**Definition 9.** [DAG-decomposition and DAG-width [BDHK06, Obd06, BDH<sup>+</sup>12]] A *DAG decomposition* of a digraph  $G$  is a pair  $\mathcal{D} = (T, \mathcal{X})$  where  $T$  is a DAG, and  $\mathcal{X} = (X_i)_{i \in V(T)}$  is a family of subsets (node bags) of  $V(G)$ , such that:

$$\bullet \bigcup_{i \in V(T)} X_i = V(G). \quad (\text{DGW-1})$$

$$\bullet \text{ For all nodes } i, j, k \in V(T), \text{ if } i \preceq j \preceq k, \text{ then } X_i \cap X_k \subseteq X_j. \quad (\text{DGW-2})$$

$$\bullet \text{ For all arcs } (i, j) \in E(T), X_i \cap X_j \text{ guards } X_{\succeq j} \setminus X_i. \text{ For any root } r \in V(T), X_{\succeq r} \text{ is guarded by } \emptyset. \quad (\text{DGW-3})$$

The width of a DAG-decomposition  $\mathcal{D} = (T, \mathcal{X})$  is defined as  $\max\{|X_i| : i \in V(T)\}$ . The *DAG-width* of  $G$ , denoted by  $\text{dgw}(G)$ , is the minimum width over all possible DAG-decompositions of  $G$ .

**Theorem 10.** For a digraph  $G$ ,  $\text{dgw}(G) \leq \text{circ}(G) + 1$ .

*Proof.* Let  $T$  be the depth-first search tree constructed in Section 1.3. We construct a DAG  $\tilde{T}$  by adding more edges to  $T$ . For a vertex  $v \in V(T)$  let  $S_v = \{u \mid \text{dfs}(u) < \text{dfs}(v) \text{ and } u \not\prec_T v\}$ . Add new edges from  $v$  to every vertex in  $S_v$ . We do this for every  $v \in V(T)$ . The graph  $\tilde{T}$  obtained in this way is a DAG.

We now define the set of node bags  $\mathcal{X} = (X_v)_{v \in V(\tilde{T})}$ . Let  $X_r = \{r\}$ . For every vertex  $v \neq r$ , we define  $X_v$  as follows:

- if there are no back edges from  $T_{\succeq v}$ , we define  $X_v = \{r\}$ .
- if there are back edges from  $T_{\succeq v}$ , let  $B$  be the set of all vertices  $b \preceq_T v$  such that there is a back edge from some vertex in  $T_{\succeq v}$  to  $b$ . Let  $b_0$  be the minimal element in  $B$  with respect to  $\preceq_T$ . Let  $X_v = \{r\} \cup \{x \mid b_0 \preceq x \preceq v\}$ . Note that  $|\{x \mid b_0 \preceq x \preceq v\}| \leq l$  and hence  $|X_v| \leq l + 1$ .

The size of each node bag is at most  $l + 1$ . We claim that  $\mathcal{D} = (\tilde{T}, \mathcal{X})$  is a DAG decomposition of  $G$ . Note that  $V(G) = V(\tilde{T})$  and  $v \in X_v$  for every vertex  $v \in V(\tilde{T})$ . Hence, (DGW-1) is satisfied.

Consider two vertices  $i \neq j$  such that  $i \in X_j$ . There exist  $b \preceq i$  and  $a \succeq j$  such that  $(a, b) \in E(G)$  is a back edge. Every vertex  $k$  such that  $i \preceq k \preceq j$  satisfies  $b \preceq k \preceq j$ , and hence by our construction  $k \in X_j$ . So, (DGW-2) is satisfied.

All the out-going edges from  $X_{\succeq j} \setminus X_i$  are either back edges (or) edges going through the root  $r$ . All the heads of the back edges from  $X_{\succeq j} \setminus X_i$  are in  $X_i \cap X_j$ . Also,  $r \in X_v$  for every  $v \in V(\tilde{T})$ . Hence,  $X_i \cap X_j$  guards  $X_{\succeq j} \setminus X_i$  and (DGW-3) is satisfied.  $\square$

### 4 Kelly-width and Circumference

Kelly-decomposition and Kelly-width were introduced by Hunter and Kreutzer [HK08].

**Definition 11.** [Kelly-decomposition and Kelly-width [HK08]] A *Kelly-decomposition* of a digraph  $G$  is a triple  $\mathcal{D} = (T, \mathcal{W}, \mathcal{X})$  where  $T$  is a DAG, and  $\mathcal{W} = (W_i)_{i \in V(T)}$  and  $\mathcal{X} = (X_i)_{i \in V(T)}$  are families of subsets (node bags) of  $V(G)$ , such that:

- $\mathcal{W}$  is a partition of  $V(G)$ . (KW-1)
- For all nodes  $i \in V(T)$ ,  $X_i$  guards  $W_{\succeq i}$ . (KW-2)
- For each node  $i \in V(T)$ , the children of  $i$  can be enumerated as  $j_1, \dots, j_s$  so that for each  $j_q$ ,  $X_{j_q} \subseteq W_i \cup X_i \cup \bigcup_{p < q} W_{\succeq j_p}$ . Also, the roots of  $T$  can be enumerated as  $r_1, r_2, \dots$  such that for each root  $r_q$ ,  $X_{r_q} \subseteq \bigcup_{p < q} W_{\succeq r_p}$ . (KW-3)

The width of a Kelly-decomposition  $\mathcal{D} = (T, \mathcal{W}, \mathcal{X})$  is defined as  $\max\{|W_i \cup X_i| : i \in V(T)\}$ . The *Kelly-width* of  $G$ , denoted by  $\text{kw}(G)$ , is the minimum width over all possible Kelly-decompositions of  $G$ .

**Theorem 12.** For a digraph  $G$ ,  $\text{kw}(G) \leq \text{circ}(G) + 1$ .

*Proof.* Let  $\tilde{T}$  be the DAG constructed in the proof of [Theorem 10](#). Let  $\mathcal{W} = (W_i)_{i \in V(T)}$  be a partition of  $V(G)$  defined as  $W_i = \{i\}$  for each  $i \in V(T)$ . We now define the set of node bags  $\mathcal{X} = (X_v)_{v \in V(\tilde{T})}$ . Let  $X_r = \emptyset$ . For every vertex  $v \neq r$ , we define  $X_v$  as follows:

- if there are no back edges from  $T_{\succeq v}$ , we define  $X_v = \{r\}$ .
- if there are back edges from  $T_{\succeq v}$ , let  $B$  be the set of all vertices  $b \preceq_T v$  such that there is a back edge from some vertex in  $T_{\succeq v}$  to  $b$ . Let  $b_0$  be the minimal element in  $B$  with respect to  $\preceq_T$ . Let  $X_v = \{r\} \cup \{x \mid b_0 \preceq x \preceq v\} \setminus v$ . Note that  $|\{x \mid b_0 \preceq x \preceq v\} \setminus v| \leq l - 1$  and hence  $|X_v| \leq l$ . We call  $b_0$  the “hook” of  $v$  and denote it by  $\text{hook}(v)$ .

The size of each node bag is at most  $l$ , so the size of each  $|W_i \cup X_i|$  is at most  $l + 1$ . We claim that  $\mathcal{D} = (\tilde{T}, \mathcal{W}, \mathcal{X})$  is a Kelly decomposition of  $G$ . By construction,  $\mathcal{W} = (W_i)_{i \in V(T)}$  is a partition of  $V(G)$  so  $\mathcal{D}$  satisfies (KW-1).

All the out-going edges from  $W_{\succeq i}$  are either back edges (or) edges going through the root  $r$ . All the heads of the back edges from  $W_{\succeq i}$  are in  $X_i$ . Also,  $r \in X_v$  for every  $v \in V(\tilde{T})$ . Hence,  $X_i$  guards  $W_{\succeq i}$  and (KW-2) is satisfied.

Recall the definition of “hook”. For a vertex  $v \in V(T)$ , if there are no back edges from  $T_{\succeq v}$ , we define  $\text{hook}(v) = v$ . For a node  $i \in V(T)$ , we enumerate the children of  $i$  as  $j_1, \dots, j_s$  such that  $\text{hook}(j_1) \succeq \text{hook}(j_2) \succeq \dots \succeq \text{hook}(j_s)$ . With this ordering, (KW-3) is satisfied.  $\square$

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